the δ's above 300 MeV depend crucially on them. This can be attacked in two ways: at moderate energies detailed knowledge of the inelastic production angular distribution might be combined with analyses similar to those of Mandelstam and Soroko. At high energies the δ's probably go to zero so that a phase shift analysis of the elastic scattering might be done solely in terms of the η's.

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Photon as a Symmetry-Breaking Solution to Field Theory. I*†

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The mechanism which guarantees the consistency of the angular-momentum conservation and commutation rules of a Lorentz-invariant theory with the requirement that the vacuum expectation of a vector operator be nonvanishing is examined in detail. A theory originally proposed by Bjorken which reproduces ordinary electrodynamics is presented in a manner which allows the calculation of the parameters of the theory. In particular the "consistency condition" is displayed and found to be quadratically, not cubically, divergent. It is shown that the original Bjorken solution occurs when the cutoff condition of the theory is taken literally. This attitude results in difficulties with current conservation and leads to transitions between the standard vacuum and anomalous degenerate states. These transitions alone, and not the ones directly involving the massless vector particles induced by the broken symmetry, are responsible for the ultimate consistency of the theory. An alternative formulation of the theory which does not take the cutoff so seriously, and hence places emphasis on the underlying operator structure rather than the perturbation Green's functions of the theory, is proposed. This presentation is essentially equivalent to the original formulation since it differs only by gauge terms. However, in this case no difficulty is encountered with current conservation and the theory is consistent in the manner required by normal formulations of the Goldstone theorem.

INTRODUCTION

IN the last few years increasing amounts of evidence have been gathered to indicate the quantum field theory has sufficient untapped potential to allow it to deal fairly simply with the vast number of observed particles. The basis of this evidence is the observation that it is probably possible for one field operator to be associated with more than one particle. This possibility was first explored by Heisenberg[1,2] and his co-workers in a series of papers on nonlinear field theory. However, Heisenberg's work involved the introduction of several concepts which are radically different from those of ordinary field theory.

Using less radical concepts developed recently to explain the phenomena of superconductivity,[3,4] it was possible for Jona-Lasinio and Nambu[5,6] to develop a nonlinear theory in which the "pion" is not introduced as a separate field but as an excitation associated with a current of the fermion field. The basic assumption that allows the occurence of the pion is that the vacuum is a degenerate state. In particular, it is assumed that the vacuum is no longer invariant under the continuous group of rotations in γ_5 space. It is then said that the γ_5 symmetry is "broken." This assumption, although it is alien to long cherished beliefs in the quantum field theory of particles, is not at all unusual in other branches of physics. The ground state of a superconductor or ordinary paramagnetism are common examples of "broken symmetries."

In this work we shall study a way of "creating" a photon by "breaking" the invariance of the vacuum under Lorentz transformations. The suggestion that a photon might be created in this manner was first made by Bjorken.[7] The possibility of generating a photon through a four-fermion interaction has also been previously examined by Heisenberg[1,2] and Birula.[8] The essential feature in the Bjorken-type theory is that the masslessness of the derived particle is associated with

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the broken symmetry. Theorems[9–11] which guarantee this point have been derived in a fairly general manner. The possible failure of the "Goldstone theorems" has been pointed out by Klein and Lee[12] using the nonrelativistic example of superconductivity in the presence of the long-range Coulomb force. In this example there are no zero-mass excitations and the consistency of the theory is associated with the slow drop off of $1/r$. This example is somewhat misleading since the mere presence of $1/r$ is a manifestation of the zero-mass photon of a fully quantized theory. Thus, in some sense a zero-mass particle is involved in this problem. There is, however, no question that the consistency of the broken-symmetry requirement has occured in a manner outside the scope of the usual Goldstone theorem. We shall show that it is only possible to understand the original Bjorken formulation of broken Lorentz symmetry by associating the consistency with spurious transitions between states built on different degenerate vacuums. Nevertheless, a massless particle is present in this example. Although the spurious states are largely associated with the nature of the approximations involved, the observation of this phenomenon suggests that the zero-mass-particle claim of the Goldstone theorem might always be correct in a fully quantized theory. However, the mechanism of its realization might be much more complicated than originally believed.

I. GENERAL CONSIDERATIONS OF CONSISTENCY

Before giving a specific example we shall examine the general properties of broken Lorentz symmetry in some detail. It is assumed that we have at our disposal a vector-fixed current $j^\mu(x)$ (not necessarily conserved) and some standard vacuum $|0\rangle$ such that

$$\langle 0|j^\mu(x)|0\rangle=\eta^\mu\neq 0. \tag{1.1}$$

Here η^μ is a constant which has the transformation properties of a Lorentz four-vector reflecting the transformation properties of $j^\mu(x)$. The assumption that η^μ is independent of the coordinate x assures us that the translational invariance properties of the vacuum are preserved. It is, however, immediately clear that Eq. (1.1) is manifestly inconsistent with the usual assumption that the vacuum is an eigenstate of any charge-conjugation operator C. Thus, if such an operator exists in the theory under consideration, the vacuum is at least twofold degenerate. It is in fact infinitely degenerate. To see this let L be the unitary operator representing a Lorentz transformation $l^{\mu\nu}$. The usual requirement that the vacuum is nondegenerate under the transformation L, that is $L^\dagger|0\rangle=|0\rangle$, and the requirement that j^μ is a four-vector, that is $L^\dagger j^\mu(x)L=j'^\mu(x)$, results in the equation

$$\eta^\mu=\langle 0|j^\mu(x)|0\rangle=\langle 0|LL^\dagger j^\mu(x)LL^\dagger|0\rangle=\langle 0|j^{\mu\prime}(x)|0\rangle.$$

Since η^μ does not depend on the coordinate x it follows that

$$\eta^\mu=\langle 0|j^{\mu\prime}(x')|0\rangle=l^{\mu\nu}\eta_\nu.$$

For a nontrivial transformation $l^{\mu\nu}$ this can only be true if $\eta^\mu=0$. We have assumed that this is not the case, and hence it follows that the vacuum cannot be an eigenstate of the transformation L. In fact, corresponding to each possible Lorentz transformation there must be a vacuum and hence the conclusion that the vacuum is infinitely degenerate. We emphasize that we have picked one of these vacuum states as a standard from which we shall construct all Green's functions. We denote this standard vacuum as $|0\rangle$.

From the above discussion it is not surprising to find that much detailed information about such a theory may be found by studying the structure of $\langle 0|L^\dagger j^\mu(x)L|0\rangle$. To this end, note that infinitesimal transformations may be characterized by $L=1+\frac{1}{2}i\delta\omega^{\mu\nu}J_{\mu\nu}$. $J_{\mu\nu}$ are the generators of Lorentz transformations and hence have the property that

$$(1/i)[J^{\mu\nu},j^\lambda(x)]=[x^\mu\partial^\nu-x^\nu\partial^\mu]j^\lambda(x)+g^{\mu\lambda}j^\nu(x)-g^{\lambda\nu}j^\mu(x).$$

Using Eq. (1.1) it follows that

$$(1/i)\langle 0|[J^{\mu\nu},j^\lambda(x)]|0\rangle=g^{\mu\lambda}\eta^\nu-g^{\lambda\nu}\eta^\mu. \tag{1.2}$$

Since the right-hand side of (1.2) does not depend on x^μ we lose no generality be setting $x^\mu=0$ on the left-hand side of this equation. It is convenient to split (1.2) into two parts. The first and most important part, reflecting the behavior of the theory under Lorentz transformations, is

$$(1/i)\langle 0|[J^{0k},j^\lambda(0)]|0\rangle=g^{0\lambda}\eta^k-g^{\lambda k}\eta^0. \tag{1.3a}$$

The second part, reflecting the behavior of the theory under ordinary spatial rotations, is

$$(1/i)\langle 0|[J^{lk},j^\lambda(0)]|0\rangle=g^{l\lambda}\eta^k-g^{\lambda k}\eta^l. \tag{1.3b}$$

Using the representation

$$J^{\mu\nu}=\int d^3y[y^\mu T^{0\nu}(y)-y^\nu T^{0\mu}(y)],$$

where $T^{\mu\nu}(y)$ is the symmetrical divergenceless energy momentum tensor, we find from (1.3a) and $P^k|0\rangle=0$ that

$$\int d^3y\, y^k i\langle 0|[T^{00}(y),j^\lambda(0)]|0\rangle=g^{0\lambda}\eta^k-g^{\lambda k}\eta^0. \tag{1.4}$$

We are then led to define the new quantity

$$C_\eta{}^{\mu\nu\lambda}(y-x)\equiv i\langle 0|[T^{\mu\nu}(y),j^\lambda(x)]|0\rangle \equiv\int d^4k\, e^{ik(y-x)}C_\eta{}^{\mu\nu\lambda}(k). \tag{1.5}$$

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Now (1.4) becomes

$$\int d^3y\, y^k C_\eta{}^{00\lambda}(y) = g^{0\lambda}\eta^k - g^{\lambda k}\eta^0, \tag{1.6a}$$

while from (1.3b) it is found that

$$\int d^3y [y^k C_\eta{}^{0l\lambda}(y) - y^l C_\eta{}^{0k\lambda}(y)] = g^{l\lambda}\eta^k - g^{\lambda k}\eta^l. \tag{1.6b}$$

$J^{\mu\nu}$ is conserved so the structure of $T^{\mu\nu}(y)$ is such that in the construction of $J^{\mu\nu}$ we can perform the integration over any surface defined by $y^0=$constant. This is the fact essential to the proof of the Goldstone theorem. However, for other considerations it is convenient to set $y^0=0$. If we set $y^0=0$ and substitute the Fourier representation of $C_\eta{}^{\mu\nu\lambda}(y)$ into Eq. (1.6a) we find, by performing an integration by parts and interchanging the $\mathbf{y}$ and k integrations, that

$$i(2\pi)^3 \int d^4k\, \delta^3(k) \frac{\partial}{\partial k_k} C_\eta{}^{00\lambda}(k) = g^{0\lambda}\eta^k - g^{\lambda k}\eta^0. \tag{1.7}$$

This can be a useful equation for certain explicit calculations, but it must be applied with extreme caution since some of the integrals encountered are ambiguously defined when the orders of integration are interchanged.

The quantity $C_\eta{}^{\mu\nu\lambda}(k)$ is symmetrical in μ and ν since $T^{\mu\nu}$ is symmetrical. That is,

$$C_\eta{}^{\mu\nu\lambda}(k) = C_\eta{}^{\nu\mu\lambda}(k). \tag{1.8a}$$

Since $T^{\mu\nu}$ is transverse we find

$$k_\mu C_\eta{}^{\mu\nu\lambda}(k) = k_\nu C_\eta{}^{\mu\nu\lambda}(k) = 0. \tag{1.8b}$$

If j^λ is a conserved current it follows that

$$k_\lambda C_\eta{}^{\mu\nu\lambda}(k) = 0. \tag{1.8c}$$

These conditions considerably restrict the tensorial form that $C_\eta{}^{\mu\nu\lambda}(k)$ may take. Because of the assumption that the broken symmetry does not affect the relativistic structure of the theory except by causing $\langle 0|j^\mu|0\rangle = \eta^\mu$, $C_\eta{}^{\mu\nu\lambda}(k)$ must be made up of three index combinations of $g^{\mu\nu}$, η^ν, and k^ν multiplied by scalar functions of these three objects. The structure of $C_\eta{}^{\mu\nu\lambda}$ holds essential information about the states of the theory. This is demonstrated by expressing $C_\eta{}^{\mu\nu\lambda}$ as

$$C_\eta{}^{\mu\nu\lambda} = i \sum_n [\langle 0|T^{\mu\nu}|\eta\rangle\langle\eta|j^\lambda|0\rangle - \langle 0|j^\lambda|n\rangle\langle n|T^{\mu\nu}|0\rangle]. \tag{1.9}$$

It is emphasized that this sum must include all possible intermediate states. In particular, besides the states built on the standard vacuum $|0\rangle$, it must include the other degenerate vacuums and the states built on them. In the special case that transitions to states not built on $|0\rangle$ are irrelevant, $C_\eta{}^{\mu\nu\lambda}(k)$ should display a standard Lehmann form and, thus representing an intermediate state of mass m, has the structure

$$C_\eta{}^{\mu\nu\lambda}(k) = \delta(k^2+m^2) C_\eta{}'^{\mu\nu\lambda}(k).$$

Generalization to many intermediate single-particle states is trivial but unnecessary. Inserting this into (1.6a) (with $y^0 \neq 0$) we have

$$\int d^3y\, y^k \int d^4k\, e^{-ik^0y^0} e^{i\mathbf{k}\cdot\mathbf{y}} \delta(k^2+m^2) C_\eta{}'^{\mu\nu\lambda}(k)$$
$$= \int d^3y\, y^k \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{y}} C_\eta{}''^{\mu\nu\lambda}(\mathbf{k}, (k^2+m^2)^{1/2}y^0) = g^{0\lambda}\eta^k - g^{\lambda k}\eta^0.$$

Here

$$C_\eta{}''^{\mu\nu\lambda}(\mathbf{k}, (k^2+m^2)^{1/2}y^0) = \frac{1}{2(\mathbf{k}^2+m^2)^{1/2}} \times [C_\eta{}'^{\mu\nu\lambda}(k)|_{k^0=(\mathbf{k}^2+m^2)^{1/2}} \exp[-i(\mathbf{k}^2+m^2)^{1/2}y^0] + C_\eta{}'^{\mu\nu\lambda}(k)|_{k^0=-(\mathbf{k}^2+m^2)^{1/2}} \exp[i(\mathbf{k}^2+m^2)^{1/2}y^0]].$$

Since the right-hand side of this equation is independent of y^0, it follows that the only terms on the left-hand side which can make nonzero contributions are those for which $m=0$. We thus conclude that if terms of the form $\delta(k^2+m^2)$ occur, they occur only with $m=0$. Thus, in a normal structure intermediate states of zero mass such that $C_\eta{}^{\mu\nu\lambda}(k) \propto \delta(k^2)$ are necessary in order to insure that (1.6) is possible for $\eta^\mu \neq 0$. This is the essence of all the general proofs of the Goldstone theorem. Before possible anomalous transitions are considered, we wish to analyze more explicitly the structure of $C_\eta{}^{\mu\nu\lambda} \propto \delta(k^2)$ when the current j^λ is conserved.

A small amount of experimentation with this restriction serves to convince one that there exist essentially only two basic three-index transverse forms. With convenient normalization these are

$$\delta(k^2)[k^\mu k^\nu k^\lambda/(\eta\cdot k)^2]\eta^2$$

and

$$\delta(k^2)\bar{g}^{\mu\nu}k^\lambda.$$

Here we have made the convenient definition

$$\bar{g}^{\mu\nu} \equiv g^{\mu\nu} - k^\mu\eta^\nu/\eta\cdot k - k^\nu\eta^\mu/\eta\cdot k.$$

From the preceding discussion it is seen that the most general form of $C_\eta{}^{\mu\nu\lambda}(k)$ consistent with (1.8) is

$$C_\eta{}^{\mu\nu\lambda}(k) = \{a\bar{g}^{\mu\nu}k^\lambda + \tfrac{1}{2}b[\bar{g}^{\mu\lambda}k^\nu + \bar{g}^{\nu\lambda}k^\mu] + c\eta^2(k^\mu k^\nu k^\lambda/(\eta\cdot k)^2)\}\delta(k^2)(\eta\cdot k)/i. \tag{1.10}$$

By virtue of relativistic invariance, the coefficients a, b, and c are functions of $(\eta\cdot k)$, η^2, and $\epsilon(k^0)$.

Many details as to the structure of the relevant parts of these three coefficients may be determined by examining Eqs. (1.6). In particular, it is clear that $C_\eta{}^{\mu\nu\lambda}$ is an odd function of η. Using this observation it is seen that a, b, and c must be even functions of $\eta\cdot k$. Since the operators $T^{\mu\nu}$ and j^λ are Hermitian, we see from Eq. (1.5) that $C_\eta^{\mu\nu\lambda *}(y-x)=C_\eta^{\mu\nu\lambda}(y-x)$. In terms of the Fourier representation of $C_\eta^{\mu\nu\lambda}$ this condition is

$$C_\eta^{\mu\nu\lambda *}(-k)=C_\eta^{\mu\nu\lambda}(k). \tag{1.11}$$

In terms of the coefficients a, b, and c, condition (1.11) is

$$a^*((\eta\cdot k)^2,\,-k,\,\eta^2)=-a((\eta\cdot k)^2,k,\eta^2),$$
$$b^*((\eta\cdot k)^2,\,-k,\,\eta^2)=-b((\eta\cdot k)^2,k,\eta^2),$$
$$c^*((\eta\cdot k)^2,\,-k,\,\eta^2)=-c((\eta\cdot k)^2,k,\eta^2).$$

Since the only function of k that remains at our disposal is $\epsilon(k^0)$ we find

$$a((\eta\cdot k)^2,k,\eta^2)=\epsilon(k^0)a_1((\eta\cdot k)^2,\eta^2)+ia_2((\eta\cdot k)^2,\eta^2), \tag{1.12a}$$

$$b((\eta\cdot k)^2,k,\eta^2)=\epsilon(k^0)b_1((\eta\cdot k)^2,\eta^2)+ib_2((\eta\cdot k)^2,\eta^2), \tag{1.12b}$$

$$c((\eta\cdot k)^2,k,\eta^2)=\epsilon(k^0)c_1((\eta\cdot k)^2,\eta^2)+ic_2((\eta\cdot k)^2,\eta^2). \tag{1.12c}$$

The coefficients a_1, a_2, b_1, b_2, c_1 and c_2 are all real.

We have now said essentially all that it is possible to say about $C_\eta^{\mu\nu\lambda}$ without introducing additional information. Before we do this it is, however, possible to make a few statements about which parts of $C_\eta^{\mu\nu\lambda}(k)$ are important for the validity of conditions (1.6). These statements are contingent on regularity assumptions and a number of calculations which will not be displayed here. The following are found through the substitution of $C_\eta^{\mu\nu\lambda}(k)$ as given by (1.10) combined with Eqs. (1.12) into (1.6a) and (1.6b).

(A) The coefficients of a_2, b_2, and c_2 make no contribution to the right-hand sides of (1.6) and hence can be ignored in our analysis.

(B) If a_1, b_1, and c_1 are expanded in a power series in $(\eta\cdot k)^2$ only the terms independent of $(\eta\cdot k)^2$ can make a nonzero contribution to the right-hand sides of (1.6). Therefore we need only consider for this purpose $a_1(0,\eta^2)$, $b_1(0,\eta^2)$, and $c_1(0,\eta^2)$.

(C) All three terms, a, b, and c, can contribute the proper structure to the right-hand sides of Eqs. (1.6).

A case that will be of prime importance occurs when $a=1/(2\pi)^3$ and $b=c=0$. In this case (1.6) are satisfied for any η. Another important case occurs through an additional restriction on $C_\eta^{\mu\nu\lambda}(k)$. This restriction is imposed because of knowledge acquired in performing model calculations. It is particularly interesting because it corresponds to calculating in an unusual gauge in ordinary electrodynamics in a constant external field. The restriction follows naturally by noting from Eq. (1.2) that

$$\frac{1}{i}\langle 0|[J^{\mu\nu},j^\lambda(x)]|0\rangle\eta_\lambda=[g^{\mu\lambda}\eta^\nu-g^{\lambda\nu}\eta^\mu]\eta_\lambda=0.$$

This suggests that

$$\eta_\lambda C_\eta^{\mu\nu\lambda}=0. \tag{1.13}$$

The condition (1.13) is the maximal restriction required to insure that Eq. (1.2) holds, but it is by no means necessary. This is an extremely prohibitive condition. Its validity requires that $a=0$ and that $b=c$. Hence, the part of Eq. (1.10) that is of interest for the validity of (1.13) becomes

$$C_\eta^{\mu\nu\lambda}(k)=[C_1\epsilon(k^0)](\delta(k^2)\eta\cdot k/i)\times[\bar{g}^{\lambda(\mu}k^{\nu)}+\eta^2(k^\mu k^\nu k^\lambda/(\eta\cdot k)^2)], \tag{1.14}$$

or more explicitly

$$C_\eta^{\mu\nu\lambda}(k)=[C_1\epsilon(k^0)(\delta(k^2)/i)\times[g^{\lambda(\mu}k^{\nu)}\eta\cdot k-\eta^\lambda k^\mu k^\nu-\eta^{(\mu}k^{\nu)}k^\lambda+\eta^2(k^\mu k^\nu k^\lambda/\eta\cdot k)].$$

This form of $C_\eta^{\mu\nu\lambda}(k)$ instantly yields an important bit of information. The term with the structure $\delta(k^2)g^{\lambda(\mu}k^{\nu)}$ shows the mixing of indices λ, μ, ν characterizing a vector intermediate-particle state of zero mass. If $C^{\mu\nu\lambda}(k)\propto\delta(k^2)k^\lambda k^\mu k^\nu$, we could not immediately conclude that a vector particle is present as this term is entirely of a scalar gauge structure.

The substitution of the structure (1.14) into Eqs. (1.6a) and (1.6b) yields a restriction on the vector η^μ. It is found that $\eta^\mu=-g^{\mu 0}\eta^0$ is impossible for $\eta^\mu\neq 0$. If $\eta^\mu=(0,\boldsymbol{\eta})$ then Eq. (1.6b) is consistent with our structure for $C_\eta^{\mu\nu\lambda}(k)$ but (1.6a) cannot be satisfied unless $\boldsymbol{\eta}=0$, contrary to hypothesis. In obtaining this result one must be exceedingly careful about exchanging orders of integration because of the singular nature of $1/\eta\cdot k$. However, we find that if $\eta^\mu\eta_\mu=0$, then both Eqs. (1.6a) and (1.6b) are consistent with $\eta^\mu\neq 0$ and that $c_1=1/(2\pi)^3$. The reason for this is fairly clear. Since we are examining a massless vector particle, the current j^μ must be identified with an electromagnetic field A^μ. Restriction (1.13) can then be restated as the gauge condition $\eta_\mu A^\mu=0$, and combined with the requirement that $\langle 0|A^\mu|0\rangle=\eta^\mu$, yields $\eta_\mu\langle 0|A^\mu|0\rangle=\eta^2=0$. The requirement $\eta^2=0$ results in the vastly simplified structure $C_\eta^{\mu\nu\lambda}(k)=[1/(2\pi)^3]\epsilon(k^0)\delta(k^2)\bar{g}^{\lambda(\mu}k^{\nu)}$+possible terms which do not contribute to the consistency of $\langle j^\mu\rangle=\eta^\mu$.

We now consider the case when the anomalous nature of the degenerate states contributes to the consistency of the broken symmetry requirement, and normal single-particle Lehmann representations are not valid. We do assume, however, that the contributions are not so anomalous as to preclude any sort of spectral representation for $C_\eta^{\mu\nu\lambda}(k)$. It is easily established that any $C_\eta^{\mu\nu\lambda}(k)$ which has the property that $C_\eta^{\mu\nu\lambda}(k)|_{\mathbf{k}\to 0}\propto\delta(k^0)$ is consistent with the time independence of $\langle 0|[J^{\mu\nu},j^\lambda(x)]|0\rangle$. As has been demonstrated, this is valid with the normal particle-spectrum assumption necessary for the proof of the presence of a zero-mass particle. However, the availability of the vector η^μ allows us easily to construct quantities which do not have a particle interpretation. In particular, note that since $\delta(\eta\cdot k)|_{\mathbf{k}\to 0}\propto\delta(k^0)$ this quantity might dominate the structure of $C_\eta^{\mu\nu\lambda}(k)$. In fact, if the restriction of current conservation is removed, it is clear that

$$C_\eta^{\mu\nu\lambda}(k)=[i/(2\pi)^3]\delta(\eta\cdot k)k^\lambda\eta^\mu\eta^\nu \tag{1.15}$$

is consistent with $\partial_\mu T^{\mu\nu}=0$ and satisfies Eqs. (1.6) for η^μ timelike or lightlike. This structure does not have any normal particle interpretation and in fact, for $\eta^\mu=-g^{\mu0}\eta^0$, gives contributions to $\langle 0|[J^{0k},j^k]|0\rangle$ which look somewhat like vacuum-vacuum transitions. This must be essentially the case, since in these broken-symmetry theories structure (1.15) may be identified as representing transitions between states built on different vacuums. We shall be able to construct a set of perturbation Green's functions that display transitions of this sort. They will, however, be directly associated with a zero-mass particle so the conclusions of the Goldstone theorem, though not the usual spectral arguments, will be correct. We suspect that this might always be the case in fully relativistic four-dimensional theories when "spurious states" are responsible for consistency of the theories.

II. THE SELF-COUPLED FERMION MODEL

We consider a charged spin-$\frac{1}{2}$ field coupled to itself by a vector Fermi interaction. This example was considered by Bjorken[7] in a less complete and somewhat misleading manner.

The Lagrangian density is taken to be

$$\mathcal{L}(x)=\tfrac{1}{2}[i\psi\cdot\beta\gamma^\mu\partial_\mu\psi-m\psi\beta\psi]+\tfrac{1}{4}g_0(\psi\beta\gamma^\mu q\psi)(\psi\beta\gamma_\mu q\psi)+J_\mu(\psi\beta\gamma^\mu q\psi). \quad (2.1)$$

We use the conventions $\beta=\gamma^0$ and $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=-2g^{\mu\nu}$. J_μ is a source current which will be set equal to zero at the end of a calculation. The field ψ is Hermitian. The use of Hermitian fields is not necessary but prevents us from falling into traps which tend to destroy the theory. The quantity q is the charge matrix such that $q=\sigma_2$. We shall use the definition

$$j^\mu\equiv\psi\beta\gamma^\mu q\psi.$$

The field equation derived from (2.1) is

$$i\beta\gamma^\mu\partial_\mu\psi-m\beta\psi+g_0j^\mu\cdot\beta\gamma_\mu q\psi+2J_\mu\beta\gamma^\mu q\psi=0. \quad (2.2)$$

Taking the transpose of Eq. (2.2) we find

$$i\partial_\mu\psi\beta\gamma^\mu+m\psi\beta-g_0j_\mu\cdot\psi\beta\gamma^\mu q-2J^\mu\psi\beta\gamma_\mu q=0. \quad (2.3)$$

From (2.2) and (2.3) it follows that $\partial_\mu j^\mu(x)=0$. We shall identify $j^\mu(x)$ with the current discussed in the first section. The symmetry is "broken" by taking

$$\eta^\mu(x)|_{J=0}\equiv\left.\frac{\langle 0\sigma_1|j^\mu(x)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}\right|_{J=0}=\eta^\mu. \quad (2.4)$$

We first study the Fermi two-point function G where

$$G(x,y)=\frac{i\langle 0\sigma_1|(\psi(x)\psi(y)\beta)_+|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}.$$

G satisfies the equation

$$\left[i\gamma^\mu\partial_\mu-m+(g_0\eta_\mu+2J_\mu)\gamma^\mu q+g_0\frac{-i\delta}{\delta J^\mu}\gamma^\mu q\right]G=-1. \quad (2.5)$$

In order to demonstrate the analogy of this theory with electrodynamics, it is convenient to define the quantity

$$\mathcal{A}^\mu\equiv g_0\eta^\mu+2J^\mu=\frac{\langle 0\sigma_1|g_0\psi\beta\gamma^\mu q\psi+2J^\mu|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}. \quad (2.6)$$

This definition is meant to be highly suggestive. The fact that it will be useful is suggested by the form of the field equation which corresponds to (2.2) in ordinary electrodynamics. This equation is

$$i\beta\gamma^\mu\partial_\mu\psi-m\beta\psi+e_0A^\mu\cdot\beta\gamma_\mu q\psi=0. \quad (2.7)$$

Comparing (2.7) to (2.2) we see that they are formally identical, within constants, if in (2.2) we make the replacement $g_0j^\mu+2J^\mu\rightarrow A^\mu$. We now define a new propagator $\mathcal{D}^{\mu\nu}(xy)$ through the equation

$$\mathcal{D}^{\mu\nu}(xy)=\frac{\delta}{\delta J_\mu(x)}\mathcal{A}^\nu(y)=2g^{\mu\nu}\delta(xy)+g_0G^{\mu\nu}(xy). \quad (2.8)$$

The quantity $G^{\mu\nu}(xy)$ is defined by

$$G^{\mu\nu}(xy)\equiv\frac{i\langle 0\sigma_1|(j^\mu(x)j^\nu(y))_+|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}-\frac{i\langle 0\sigma_1|j^\mu(x)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}\frac{\langle 0\sigma_1|j^\nu(y)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}. \quad (2.9)$$

It is our intention to identify $\mathcal{D}^{\mu\nu}(x,y)$ with the photon propagator of ordinary electrodynamics. We shall in fact show the consistency of this identification by observing, in the case where $J=0$ and $\mathcal{D}^{\mu\nu}(xy)=\int e^{ik\cdot(x-y)}\mathcal{D}^{\mu\nu}(k)$, that $\mathcal{D}^{\mu\nu}(k)$ has a pole for $k^2=0$ and that the residue of this pole has a part that is proportional to $g^{\mu\nu}$. By the chain rule,

$$\frac{\delta}{\delta J^\mu(x)}=\frac{\delta\mathcal{A}_\nu(y)}{\delta J^\mu(x)}\frac{\delta}{\delta\mathcal{A}_\nu(y)}=\mathcal{D}_{\mu\nu}(x,y)\frac{\delta}{\delta\mathcal{A}_\nu(y)}. \quad (2.10)$$

Therefore, using (2.6) and (2.10), Eq. (2.5) may be rewritten in the suggestive form

$$\left[i\gamma^\mu\partial_\mu-m+\mathcal{A}^\mu\gamma_\mu q-g_0i\gamma_\mu q\mathcal{D}^{\mu\nu}\frac{\delta}{\delta\mathcal{A}^\nu}\right]G=-1. \quad (2.11)$$

As in ordinary electrodynamics, it is convenient to define a vertex function by

$$q\Gamma(\mu,xy)\equiv\delta G^{-1}(xy)/\delta\mathcal{A}^\mu(\mu). \quad (2.12)$$

The appropriateness of (2.12) is confirmed by noticing that in lowest order for G^{-1} that $q\Gamma^\mu=\gamma^\mu q$. With this new definition we may write Eq. (2.11) as

$$[i\gamma^\mu\partial_\mu - m + \mathcal{A}^\mu\gamma_\mu q - g_0 i\gamma_\mu q\mathfrak{D}^{\mu\nu}Gq\Gamma_\nu]G = -1. \quad (2.13)$$

Corresponding to (2.13) it is found in ordinary electrodynamics that

$$[i\gamma^\mu\partial_\mu - m + e_0\mathcal{A}_e{}^\mu\gamma_\mu q - e_0 i\gamma_\mu q\mathfrak{D}_e{}^{\mu\nu}Gq\Gamma_\nu]G = -1. \quad (2.14)$$

Here $\mathcal{A}_e{}^\mu = \langle 0\sigma_1|A^\mu|0\sigma_2\rangle / \langle 0\sigma_1|0\sigma_2\rangle$. The similarity between (2.13) and (2.14) is striking. When we have demonstrated that $\mathfrak{D}^{\mu\nu}$ has the correct properties we shall have created, as a result of this similarity, an electrodynamic theory from the Lagrangian (2.1). Note that Eq. (2.14) is exact. There is one important difference between Eqs. (2.13) and (2.14). In Eq. (2.14), when all external sources are turned off, $\mathcal{A}_e{}^\mu = 0$ by virtue of charge-conjugation invariance. On the other hand, in Eq. (2.13) when $J^\mu = 0$, $\mathcal{A}^\mu = g_0\eta^\mu$. Thus, in the absence of external sources, Eq. (2.13) has the behavior of electrodynamics in a constant external field $\mathcal{A}^\mu = g_0\eta^\mu = (\partial/\partial x_\mu)(g_0\eta^\alpha x_\alpha)$. A constant A^μ field in electrodynamics can have no physical effect since it produces no electric or magnetic fields. However, once such a constant field is introduced into a theory, it must be handled in a careful manner in order to avoid destroying the consistency of the theory.

In order to construct this theory explicitly we shall examine just the lowest approximations and show that they have the correct structure. Iteration of these approximations through the definitions and equations above serves to establish the equivalence, in all orders, of our theory to electrodynamics. It is emphasized that this iteration must be done with great care in order to maintain relation (2.4).

The first approximation for the function G is obtained from (2.11) by dropping the $\delta G/\delta\mathcal{A}^\nu$ term. We then find that

$$[i\gamma^\mu\partial_\mu - m + 2J^\mu\gamma_\mu q + g_0\eta^\mu\gamma_\mu q]G = -1. \quad (2.15)$$

If we set $J^\mu = 0$, G has the representation

$$G(x,y) = \int d^4p\, e^{ip\cdot(x-y)}G(p).$$

From Eq. (2.15) it follows that

$$G(p) = \{(2\pi)^4[\gamma\cdot p - g_0 q\gamma\cdot\eta + m]\}^{-1}. \quad (2.16)$$

It is often useful to expand $G(p)$ in terms of the eigenvalues $+1$ and -1 of q. We therefore make the definitions

$$G_+(p) = [(2\pi)^4(\gamma\cdot p - g_0\gamma\cdot\eta + m)]^{-1} \quad (2.17a)$$

and

$$G_-(p) = [(2\pi)^4(\gamma\cdot p + g_0\gamma\cdot\eta + m)]^{-1}. \quad (2.17b)$$

It can then be seen that

$$G(p) = \tfrac{1}{2}(G_+(p) + G_-(p)) + q[\tfrac{1}{2}(G_+(p) - G_-(p))]. \quad (2.18)$$

Combining the denominators that occur in expression (2.18) we find the other convenient representation for $G(p)$. This is

$$\begin{aligned} G(p) = &-(2\pi)^{-4}[(p^2+m^2+g_0{}^2\eta^2)^2 - 4g_0{}^2(\eta\cdot p)^2]^{-1} \\ &\times\{(\gamma\cdot p - m)(p^2+m^2+g_0{}^2\eta^2) - 2g_0{}^2(\gamma\cdot\eta)(\eta\cdot p) \\ &+ g_0 q(-\gamma\eta[p^2+m^2+g_0{}^2\eta^2] \\ &+ (\gamma\cdot p - m)2(\eta\cdot p))\}. \quad (2.19) \end{aligned}$$

It is necessary that the condition (2.4) on the current be satisfied. In terms of $G(p)$ this condition is

$$\eta^\mu = -\frac{1}{i}\int d^4p\, \mathrm{tr}\, q\gamma^\mu G(p). \quad (2.20)$$

Using Eq. (2.19), Eq. (2.20) may be written as

$$\eta^\mu = \eta_\nu \frac{8g_0}{i}\int \frac{d^4p}{(2\pi)^4} \frac{g^{\nu\mu}(p^2+m^2+g_0{}^2\eta^2) - 2p^\nu p^\mu}{(p^2+m^2+g_0{}^2\eta^2)^2 - 4g_0{}^2(\eta\cdot p)^2}. \quad (2.21)$$

On inspection of Eq. (2.21) one sees that the integral on the right-hand side is quadratically divergent. Therefore, in order for condition (2.4) to make sense, this integral must be cut off in some manner. For most of our discussions it will be unnecessary to make the exact form of the cutoff explicit. We require that the nature of the cutoff is such that it does not appear in the tensor structure of integrals and hence does not interfere with the relativistic structure of our theory.

With this in mind the integral appearing in (2.21) may be written as

$$\frac{8g_0}{i(2\pi)^4}\int d^4p \frac{g^{\nu\mu}(p^2+m^2+g_0{}^2\eta^2) - 2p^\mu p^\nu}{(p^2+m^2+g_0{}^2\eta^2)^2 - 4g_0{}^2(\eta\cdot p)^2} = \alpha(\eta^2,\Lambda)g^{\mu\nu} + \gamma(\eta^2,\Lambda)\eta^\mu\eta^\nu. \quad (2.22)$$

In (2.22), Λ is the parameter characterizing the cutoff. Inserting (2.22) into (2.21) we obtain the condition

$$\eta^\mu = \eta^\mu[\alpha(\eta^2,\Lambda) + \eta^2\gamma(\eta^2,\Lambda)]. \quad (2.23)$$

This has a trivial solution $\eta^\mu = 0$. However, the requirement of the broken symmetry is that $\eta^\mu \neq 0$. Using this requirement, Eq. (2.23) reduces to the condition

$$1 = \alpha(\eta^2,\Lambda) + \eta^2\gamma(\eta^2,\Lambda). \quad (2.24)$$

Before the coefficients α and γ are determined we wish to examine Eq. (2.20) using (2.23) as a starting point. In this case

$$\begin{aligned} \eta^\mu &= i\int d^4p[\mathrm{tr}\gamma^\mu G_+(p) - \mathrm{tr}\gamma^\mu G_-(p)] \\ &= \frac{i}{(2\pi)^4}\left[\int d^4p\, \mathrm{tr}\gamma^\mu \frac{1}{[\gamma\cdot p - g_0\gamma\eta + m]} - \int d^4p\, \mathrm{tr}\gamma^\mu \frac{1}{[\gamma\cdot p + g_0\gamma\eta + m]}\right]. \quad (2.25) \end{aligned}$$

In the above representation it is extremely tempting to translate the first integral that appears by making the substitution $p-g_0\eta \rightarrow p'$ and to translate the second integral by making the substitution $p+g_0\eta \rightarrow p'$. If this is done with abandon we get $\eta^\mu=0$. However, it must be borne in mind that the range of integration on p is not infinite but is restricted by the cutoff Λ. Therefore the translation suggested above is reflected in the limits of integration and to avoid obtaining $\eta^\mu=0$ these limits must be finite. Thus, the cutoff Λ must be taken fairly seriously. Just how seriously it should be taken leads us to some interesting problems. To avoid difficulties we must be exceedingly careful about translating integrals that are divergent when $\Lambda \rightarrow \infty$. We shall find in the most realistic formulation of the theory that the above integral is the only one in which we encounter any difficulty. With the assumption that Λ/m is very large it is always possible to translate integrals which are finite in the limit $\Lambda \rightarrow \infty$ without regard to the effect on the limits of integration, since for large Λ/m these integrals are independent of Λ.

As an application of this we note that since the quantity

$$\omega_\pm{}^\mu(g_0\eta) \equiv i\int d^4p \, \mathrm{tr}\gamma^\mu G_\pm(p)$$

goes as Λ^3 for $\Lambda \rightarrow \infty$, it follows that

$$\omega_{f\pm}{}^\mu(g_0\eta) = i\int d^4p \, \mathrm{tr}\gamma^\mu \left\{ G_\pm(p) - [G_\pm(p)]_{g_0=0} - \left(\frac{d}{dg_0}G_\pm(p)\right)_{g_0=0} g_0 - \frac{1}{2}\left(\frac{d^2}{dg_0{}^2}G_\pm(p)\right)_{g=0} g_0{}^2 - \frac{1}{3!}\left(\frac{d^3}{dg_0{}^3}G_\pm(p)\right)_{g=0} g_0{}^3 \right\} \quad (2.26)$$

is finite as $\Lambda \rightarrow \infty$. It follows that the integral involved in (2.26) may be translated without regard to the cutoff. Therefore $\omega_{f+}{}^\mu = \omega_{f-}{}^\mu = 0$. Consequently

$$i\int d^4p [\mathrm{tr}\gamma^\mu G_+(p) - \mathrm{tr}\gamma^\mu G_-(p)] = \omega_+{}^\mu(g_0\eta) - \omega_-{}^\mu(g_0\eta)$$
$$= i\int d^4p \left\{ 2g_0\left[\frac{d}{dg_0}G_+(p)\right]_{g_0=0} + \frac{2g_0{}^3}{3!}\left(\frac{d^3}{dg_0{}^3}G_+(p)\right)_{g_0=0} \right\}.$$

We have found the significant result that the quantity $\omega_+{}^\mu(g_0\eta) - \omega_-{}^\mu(g_0\eta)$ will depend only on g_0 and $g_0{}^3$ and consequently only on $g_0\eta^\mu$ and $g_0{}^3\eta^2\eta^\mu$. This information is most conveniently put to use by noting that from the right-hand side of Eq. (2.21) we have

$$\omega_+{}^\mu(g_0\eta) - \omega_-{}^\mu(g_0\eta) = \frac{\eta_\nu 8g_0}{(2\pi)^4 i}\int \frac{d^4p \, g^{\nu\mu}(p^2+m^2+g_0{}^2\eta^2) - 2p^\mu p^\nu}{(p^2+m^2+g_0{}^2\eta^2)^2 - 4g_0{}^2(\eta\cdot p)^2}.$$

To obtain the behavior of this expression we expand the denominator occurring in the integrand of the right-hand side. We find

$$\frac{\eta_\nu 8g_0}{(2\pi)^4 i}\int d^4p \frac{g^{\mu\nu}\{[p]+g_0{}^2\eta^2\} - 2p^\mu p^\nu}{\{[p]+g_0{}^2\eta^2\}^2 - 4g_0{}^2(\eta\cdot p)^2}$$
$$= \frac{\eta_\nu 8g_0}{(2\pi)^4 i}\int d^4p \frac{1}{[p]^2}\left[1 + \frac{4g_0{}^2(\eta\cdot p)^2}{[p]^2} - \frac{2g_0{}^2\eta^2}{[p]}\right] \times \{g^{\mu\nu}([p]+g_0{}^2\eta^2) - 2p^\mu p^\nu\}. \quad (2.27)$$

Terms of order $g_0{}^4$ and higher have been discarded since by the above discussion they make no contribution. We have introduced the space-saving notation $p^2+m^2=[p]$. To evaluate (2.27) we shall demonstrate our method of evaluating the integrals encountered.

The basic quadratically divergent integral that occurs in the theory is

$$Q(\Lambda,m^2) \equiv \frac{8g_0}{(2\pi)^4 i}\int \frac{d^4p}{p^2+m^2}. \quad (2.28)$$

The basic logarithmically divergent quantity is

$$L(\Lambda,m^2) \equiv \frac{8g_0}{(2\pi)^4 i}\int \frac{d^4p}{(p^2+m^2)^2}. \quad (2.29)$$

The basic finite integral that occurs is

$$\frac{F(\Lambda,m^2)}{m^2} = \frac{8g_0}{(2\pi)^4 i}\int \frac{d^4p}{(p^2+m^2)^3}. \quad (2.30)$$

It is possible to express all integrals that occur in terms of Q, L, and F without ever explicitly calculating these quantities.

It is, however, of some convenience to choose a particular cutoff scheme and to calculate Q, L, and F in this scheme. We choose to use an invariant 4-momentum cutoff so that the upper limit in any of our integrals is specified by $p_{\max}{}^2=\Lambda^2$ in Euclidian four-space. We then find that

$$Q = \frac{g_0}{2\pi^2}\left[\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right]_{\Lambda\rightarrow\infty} = \frac{g_0\Lambda^2}{2\pi^2}, \quad (2.31)$$

while

$$L = \frac{g_0}{2\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{1}{1+m^2/\Lambda^2}\right]_{\Lambda\rightarrow\infty} = \frac{g_0}{\pi^2}\ln\frac{\Lambda}{m} \quad (2.32)$$

and

$$\frac{F}{m^2} = \frac{g_0}{4\pi^2}\left[\frac{1}{m^2}\left(\frac{1}{m^2/\Lambda^2+1}\right) - \frac{1}{\Lambda^2}\frac{1}{(1+m^2/\Lambda^2)^2}\right]_{\Lambda\rightarrow\infty} = \frac{g_0}{4\pi^2 m^2}. \quad (2.33)$$

It is now elementary to evaluate the other integrals which appear. As an example, consider

$$\frac{8g_0}{(2\pi)^4 i}\int d^4p\frac{p^\mu p^\nu}{(p^2+m^2)^3}.$$

By the postulated nature of the cutoff and relativistic invariance it follows that

$$\frac{8g_0}{(2\pi)^4 i}\int d^4p\frac{p^\mu p^\nu}{(p^2+m^2)^3}=\frac{g^{\mu\nu}}{4}X.$$

The quantity X is determined by noting that

$$\frac{8g_0}{(2\pi)^4 i}\int d^4p\frac{p^2}{(p^2+m^2)^3}=\frac{8g_0}{(2\pi)^4 i}\int d^4p\frac{1}{(p^2+m^2)^2}-\frac{8g_0m^2}{(2\pi)^4 i}\int\frac{d^4p}{(p^2+m^2)^3}=\frac{g^\mu{}_\mu X}{4}=X$$

or $X=L-F$. We have thus found

$$\frac{8g_0}{(2\pi)^4 i}\int d^4p\frac{p^\mu p^\nu}{(p^2+m^2)^3}=\frac{g^{\mu\nu}}{4}[L-F]. \quad (2.34a)$$

For the time being we do not drop F because it is small relative to L. This sort of approximation has profound effect on the structure of the theory and will be discussed in detail. Using the same methods it is found that

$$\frac{8g_0}{(2\pi)^4 i}\int d^4p\frac{p^\mu p^\nu p^\alpha p^\beta}{(p^2+m^2)^4}=[g^{\mu\nu}g^{\alpha\beta}+g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}]\frac{1}{24}\left(L-\frac{5}{3}F\right). \quad (2.34b)$$

Inserting the results of Eq. (2.34) into (2.27) we find that

$$i\int d^4p\ \mathrm{tr}\gamma^\mu qG(p)=\eta^\mu(\tfrac{1}{2}Q+\tfrac{1}{2}m^2L-\tfrac{1}{3}g_0{}^2\eta^2F). \quad (2.35)$$

The condition (2.20) thus becomes

$$1=\tfrac{1}{2}Q+\tfrac{1}{2}m^2L-\tfrac{1}{3}g_0{}^2\eta^2F. \quad (2.36)$$

In the case that the cutoff Λ is immense, (2.36) has the form $1=g_0\Lambda^2/4\pi^2$.

We are now prepared to examine the propagator $\mathfrak{D}^{\mu\nu}$. To this end we study the function $G^{\mu\nu}(x,y)$ as defined by Eq. (2.9). Using the fact that J^μ is the source generating j_μ we find the useful representations

$$G^{\mu\nu}(z,y)=i\left[\frac{-i\delta}{\delta J_\mu(z)}\eta^\nu(y)\right]=-\left[\frac{-i\delta}{\delta J_\mu(z)}\mathrm{tr}[G(x,y)\gamma^\nu q]\right]_{x\to y}. \quad (2.37)$$

Manipulations are somewhat simplified if we define a new quantity G^μ as

$$G^\mu(z;x,y)\equiv\frac{-i\delta}{\delta J_\mu(z)}G(x,y).$$

Equation (2.37) becomes, with the aid of this definition,

$$G^{\mu\nu}(z,y)=-\mathrm{tr}[G^\mu(z;x,x)\gamma^\nu q]. \quad (2.38)$$

Since $G^{-1}G=1$ it follows that $(-i\delta/\delta J^\mu)(G^{-1}G)=0$ and thus we have

$$G^\mu(z;x,y)=-G(x\xi)[(-i\delta/\delta J_\mu(z))G^{-1}(\xi\xi')]G(\xi',y). \quad (2.39)$$

To insure the consistency of our approximations we use G^{-1} as determined by Eq. (2.15). Displaying all the indices this is

$$G^{-1}(\xi\xi')=[i\gamma^\alpha\partial_\alpha\delta(\xi-\xi')+g_0\eta_\alpha(\omega)\gamma^\alpha q\delta(\omega-\xi)\delta(\xi-\xi')+2J_\alpha(\omega)\gamma^\alpha q\delta(\omega-\xi)\delta(\xi-\xi')-m\delta(\xi-\xi')]. \quad (2.40)$$

Inserting (2.40) into (2.39) and using (2.38) we find that

$$G^\mu(z;x,y)=-iG(x\xi)[2\gamma^\mu q\delta(z-\xi)+g_0G^{\mu\alpha}(z,\xi)\gamma_\alpha q]G(\xi,y). \quad (2.41)$$

Inserting (2.41) into (2.38) and setting $J=0$ we find

$$[g_\lambda{}^\nu\delta(y-\xi)-g_0 i\ \mathrm{tr}\gamma^\nu G(y-\xi)\gamma_\lambda G(\xi-y)]G^{\lambda\mu}(\xi-z)=2i\ \mathrm{tr}[\gamma^\nu G(y-z)\gamma^\mu G(z-y)]. \quad (2.42)$$

Using the representation $G^{\lambda\mu}(z-\xi)=\int d^4p\ e^{ip\cdot(z-\xi)}\times G^{\lambda\mu}(p)$, Eq. (2.42) becomes

$$\left[g_\lambda{}^\nu-(2\pi)^4 ig_0\int d^4p\ \mathrm{tr}[\gamma^\nu G(p)\gamma_\lambda G(p+k)]\right]G^{\lambda\mu}(k)=2i\int d^4p\ \mathrm{tr}[\gamma^\nu G(p)\gamma^\mu G(p+k)]. \quad (2.43)$$

Using the Fourier representation of the photon propagator it follows from Eq. (2.8) that

$$\mathfrak{D}^{\lambda\mu}(k)=[2g^{\mu\lambda}/(2\pi)^4]+g_0G^{\lambda\mu}(k). \quad (2.44)$$

If we solve (2.43) for $G^{\lambda\mu}(k)$ and insert this result into Eq. (2.44), we find

$$\mathfrak{D}_{\nu\lambda}{}^{-1}(k)=\tfrac{1}{2}(2\pi)^4\left[g_{\nu\lambda}-(2\pi)^4 ig_0\ \mathrm{tr}\int d^4p\times\gamma_\nu G(p)\gamma_\lambda G(p+k)\right]. \quad (2.45)$$

Before we can invert $\mathfrak{D}_{\nu\lambda}{}^{-1}(k)$, it is necessary to evaluate

$$\frac{(2\pi)^4}{i}g_0\ \mathrm{tr}\int d^4p[\gamma_\nu G(p)\gamma_\lambda G(p+k)]\equiv\pi_{\nu\lambda}(k). \quad (2.46)$$

$\pi_{\nu\lambda}(k)$ is quadratically divergent as $\Lambda\to\infty$. Since $\pi_{\nu\lambda}(k)$ is even in k it follows that the quantity

$$\pi_f^{\nu\lambda}(k)=\pi^{\nu\lambda}(k)-\pi^{\nu\lambda}(0)-[(\partial/\partial k^l)(\partial/\partial k^m)\pi^{\nu\lambda}(k)]_{k=0}\tfrac{1}{2}k^lk^m \quad (2.47)$$

is finite. It is our intention to find $\pi^{\nu\lambda}(k)$ through Eq. (2.47) rewritten in the form

$$\pi^{\nu\lambda}(k)=\pi_f^{\nu\lambda}(k)+\pi^{\nu\lambda}(0)+[(\partial/\partial k^l)(\partial/\partial k^m)\pi^{\nu\lambda}(k)]_{k=0}\tfrac{1}{2}k^lk^m. \quad (2.48)$$

We shall first find $\pi_f^{\nu\lambda}(k)$. Using Eq. (2.18) it is seen that, as an alternate expression to (2.48), we may write

$$\pi_{\nu\lambda}(k)=\frac{(2\pi)^4}{i}g_0\,\mathrm{tr}\int d^4p[\gamma_\nu G_+(p)\gamma_\lambda G_+(p+k)+\gamma_\nu G_-(p)\gamma_\lambda G_-(p+k)]. \quad (2.49)$$

It is easily found that to determine $\pi_f^{\nu\lambda}(k)$ we need only calculate the terms of order k^4 and greater of

$$\frac{2g_0}{i(2\pi)^4}\int d^4p\,\mathrm{tr}\left[\gamma_\nu\frac{1}{(\gamma\cdot p+m)}\gamma_\lambda\frac{1}{\gamma\cdot(p+k)+m}\right].$$

This function is the "photon mass" of ordinary electrodynamics. This is a well known function and it is cataloged in any textbook. It is found that

$$\pi_f^{\nu\lambda}(k)=(g^{\nu\lambda}k^2-k^\nu k^k)I(k^2)2g_0, \quad (2.50)$$

where

$$I(k^2)=\frac{1}{4\pi^2}\frac{k^2}{3}\int_{4m^2}^{\infty}\frac{(1+2m^2/\kappa^2)(1-4m^2/\kappa^2)^{1/2}}{\kappa^2[\kappa^2+k^2-i\epsilon]}d\kappa^2$$

is the function appearing in ordinary electrodynamics.

Next we shall evaluate

$$\pi^{\nu\lambda}(0)=\frac{(2\pi)^4}{i}g_0\int d^4p\,\mathrm{tr}\gamma^\nu G(p)\gamma^\lambda G(p).$$

This task is made quite simple by noting that from Eq. (2.16) it follows that

$$\partial G(p)/\partial p^\mu=-(2\pi)^4G(p)\gamma^\mu G(p)=-(1/g_0)q(\partial/\partial\eta^\mu)G(p).$$

Inserting this into the form above for $\pi^{\nu\lambda}(0)$ results in

$$\pi^{\nu\lambda}(0)=\frac{1}{i}\frac{\partial}{\partial\eta^\lambda}\int d^4p\,\mathrm{tr}q\gamma^\nu G(p). \quad (2.51)$$

Using (2.51), Eqs. (2.35) and (2.36), we have

$$\pi^{\nu\lambda}(0)=-g^{\nu\lambda}+\tfrac{2}{3}Fg_0^2\eta^\nu\eta^\lambda. \quad (2.52)$$

It is convenient to introduce a new constant $C\equiv\frac{2}{3}Fg_0^2$ in which case (2.52) becomes

$$\pi^{\nu\lambda}(0)=-g^{\nu\lambda}+C\eta^\nu\eta^\lambda. \quad (2.53)$$

To complete the analysis it is necessary to evaluate the parts of $\pi^{\nu\lambda}(k)$ quadratic in k. The quantity $\partial^2\pi^{\nu\lambda}(k)/\partial k^l\partial k^m|_{k=0}$ behaves as $\ln(\Lambda/m)$ as $\Lambda\to\infty$. The quantity

$$\left.\frac{\partial^2\pi^{\nu\lambda}(k)}{\partial k^l\partial k^m}\right|_{k=0}-\left[\frac{\partial^2\pi^{\nu\lambda}(k)}{\partial k^l\partial k^m}\right]_{k=\eta=0}\equiv\left[\left.\frac{\partial^2\pi^{\nu\lambda}(k)}{\partial k^l\partial k^m}\right|_{k=0}\right]_{\mathrm{finite}}$$

is finite as $\Lambda\to\infty$ and hence we may translate η in the integral representation of

$$\left[\left.\frac{\partial^2\pi^{\nu\lambda}(k)}{\partial k^l\partial k^m}\right|_{k=0}\right]_{\mathrm{finite}}$$

to find that this quantity is zero. We therefore conclude that $\partial^2\pi^{\nu\lambda}(k)/\partial k^l\partial k^m|_{k=0}$ cannot depend on η. It follows that

$$(\partial\pi^{\nu\lambda}(k)/\partial k^l\partial k^m)|_{k=0}\tfrac{1}{2}k^lk^m=[\tfrac{1}{3}L+4F/9][g^{\nu\lambda}k^2-k^\nu k^\lambda]+(\tfrac{1}{3}F)k^\nu k^\lambda. \quad (2.54)$$

Combining (2.50), (2.53), and (2.54) we find that

$$\pi_f^{\nu\lambda}(k)=-g^{\nu\lambda}+(g^{\nu\lambda}k^2-k^\nu k^\lambda)[2g_0I(k^2)+\tfrac{1}{3}L+4F/9]+\tfrac{1}{3}F_k{}^\nu k^\lambda+C\eta^\nu\eta^\lambda, \quad (2.55)$$

and hence

$$(\mathfrak{D}^{-1})^{\nu\lambda}(k)=\tfrac{1}{2}(2\pi)^4[(g^{\nu\lambda}k^2-k^\nu k^\lambda)\bar{I}(k^2)+\tfrac{1}{3}Fk^\nu k^\lambda+C\eta^\nu\eta^\lambda]. \quad (2.56)$$

Here $\bar{I}(k^2)\equiv 2g_0I(k^2)+\frac{1}{3}L+4F/9$. We only invert Eq. (2.56) in the case that Λ is very large and hence $L\gg F$. Then the term $\frac{1}{3}Fk^\nu k^\lambda$ is dominated by $\bar{I}(k^2)k^\nu k^\lambda$ in this case and we may drop it, leaving

$$(\mathfrak{D}^{-1})_{\nu\lambda}(k)=\tfrac{1}{2}(2\pi)^4[(g_{\nu\lambda}k^2-k_\nu k_\lambda)\bar{I}(k^2)+C\eta_\nu\eta_\lambda]. \quad (2.57)$$

This is the form considered by Bjorken. We, however, have explicitely calculated all the coefficients in our scheme and can make the observation that C is exceedingly small relative to $\bar{I}(k^2)$ when g_0 is small and Λ large. This term is present in virtue of taking the cutoff very seriously, and there is some question as to whether this should be done. We shall, therefore, consider two possibilities, one where C is taken seriously, and the other corresponding to normal procedures where $C=0$. In the first case we find

$$\mathfrak{D}^{\nu\lambda}(k)=\frac{2}{(2\pi)^4}\left[\frac{1}{k^2\bar{I}(k^2)}\left[\bar{g}^{\nu\lambda}+\eta^2\frac{k^\nu k^\lambda}{(\eta\cdot k)^2}\right]+\frac{1}{C}\frac{k^\nu k^\lambda}{(\eta\cdot k)^2}\right]. \quad (2.58)$$

This is the result of Bjorken with explicit coefficients and generalized to any η^μ. Within gauge type terms this looks as a photon propagator should, as promised earlier. It is understood that the singularity $1/k^2I(k^2)$ is defined in the usual way by adding a small imaginary part to k^2.

Any singularities in the neighborhood of $\eta\cdot k=0$ are handled by the identification $1/(\eta\cdot k)^2\to p[1/(\eta\cdot k)^2]$ and $1/\eta\cdot k\to p(1/\eta\cdot k)$. This procedure insures that $1/\eta\cdot k$ terms have no imaginary parts so that

$$\mathrm{Im}\,\mathfrak{D}^{\nu\lambda}(k)=\frac{\pi}{I(0)}\delta(k^2)\left[\bar{g}^{\nu\lambda}+\frac{\eta^2k^\nu k^\lambda}{(\eta\cdot k)^2}\right];\qquad(-k^2<4m^2).\quad(2.59)$$

It follows that

$$k_\nu\,\mathrm{Im}\,\mathfrak{D}^{\nu\lambda}(k)=0\,,\quad(-k^2<4m^2).\qquad(2.60)$$

The comparison of $\mathfrak{D}^{\nu\lambda}(k)$ to the photon propagator in ordinary electrodynamics is facilitated by fully writing out (2.58). We then have

$$\frac{1}{g_0}\left[\frac{6g_0}{L}\right]\frac{[g^{\nu\lambda}-k^\nu\eta^\lambda/\eta\cdot k-\eta^\nu k^\lambda/\eta\cdot k+\eta^2k^\nu k^\lambda/(\eta\cdot k)^2]}{(2\pi)^4[1+(6g_0/L)I(k^2)]k^2}+\frac{2}{(2\pi)^4C}\frac{k^\nu k^\lambda}{(\eta\cdot k)^2}.\qquad(2.61)$$

Aside from gauge terms and normalization, we see that the above propagator is identical to the second-order photon propagator of ordinary electrodynamics if we make the identification

$$\alpha=6g_0/L.\qquad(2.62)$$

Assuming Λ/m is large this becomes, through Eq. (2.32),

$$\alpha=6\pi^2/\ln(\Lambda/m)$$

or

$$\Lambda/m\sim e^{700}.$$

This assures that Λ is huge. Inserting this into the condition (2.36) for Λ huge we find that

$$g_0m^2/4\pi^2=e^{-1400}.$$

This is ample assurance that g_0 is small for any reasonable m.

The propagating parts of this form of $\mathfrak{D}^{\mu\lambda}(k)$, that is, $\mathfrak{D}_p{}^{\nu\lambda}(k)\equiv\mathfrak{D}^{\nu\lambda}(k)-2k^\nu k^\lambda/(2\pi)^4C(\eta\cdot k)^2$, have the useful property that $\eta_\nu\mathfrak{D}_p{}^{\nu\lambda}(k)=0$. This shows, as was pointed out in the first section, that it is appropriate for the comparison to electrodynamics to take $\eta^2=0$. The terms proportional to $k^\nu k^\lambda/(\eta\cdot k)^2$ are particularly interesting. Although they are in the excitation spectrum of j^μ, they are not in any normal sense one-particle states. They appear explicitly because of the introduction of nonzero η^μ into the theory and may be interpreted as representing transitions between states built on different, degenerate vacua. Part of the strange nature of these "nonpropagating terms" is due to the fact that, though the immediate cause of their origin is the broken Lorentz symmetry, they represent, in addition, an interference with the conservation of the current j^μ. This is suggested by examining the form of the current-current commutation relations which follow from the real part of this Bjorken form of the photon propagator. This follows more directly by noting that the behavior given by (2.60) is essential since

$$\mathrm{Im}\,\mathfrak{D}^{\mu\nu}(x-y)=\mathrm{Im}\,i\langle|(j^\mu(x)j^\nu(y))_+|\rangle=i\langle0|\{j^\mu(x),j^\nu(y)\}|0\rangle$$

and hence $k_\mu\,\mathrm{Im}\,\mathfrak{D}^{\mu\nu}(k)=0$.

Note that if $-k^2>4m^2$ then (2.50) is no longer a correct representation for $\mathrm{Im}\,\mathfrak{D}^{\nu\lambda}(k)$. It is easily verified that the correct representation does not satisfy (2.60). Therefore the theory constructed in this manner is not consistent with current conservation for large $-k^2$. If the cutoff is on the order of m this would not be particularly disturbing. However, we have demonstrated that to duplicate electrodynamics it is necessary for Λ/m to be very large. It should be emphasized that this weird behavior is not particularly catastrophic. Once the field equations are used to derive the Green's function equations, we may dispense with them and consider the Green's functions on their own merits. Further, the offensive terms are all gauge terms, and hence are irrelevant in physically significant measurements which should be gauge invariant. Therefore, we feel that in this sense the Bjorken formulation of the theory is plausible.

We may, however, dispense with all these difficulties at once by deciding not to take the bounded term $C\eta^\mu\eta^\lambda$ seriously in (2.57) but to regard it as an anomaly of a cutoff procedure which would not be realized in a better calculation. In this case we find that

$$(\mathfrak{D}^{-1})_{\nu\lambda}(k)=\tfrac{1}{2}(2\pi)^4[(g_{\nu\lambda}k^2-k_\nu k_\lambda)\bar{I}(k^2)]$$

and hence

$$\mathfrak{D}_{\nu\lambda}(k)=\frac{2[g_{\nu\lambda}-k_\nu k_\lambda/k^2]}{(2\pi)^4k^2\bar{I}(k^2)}.\qquad(2.63)$$

This is the second-order Lorentz gauge photon propagator. It has the property $k^\nu\mathfrak{D}_{\nu\lambda}(k)=0$. Consequently, we do not encounter the previous difficulty with current conservation. We do this, however, by forfeiting a cutoff Green's function theory which can be regarded fairly seriously. Both forms will be retained for $\mathfrak{D}^{\nu\lambda}(k)$ for use in examining the consistency of the theory. This check is highly gauge-dependent, and it will be very interesting to see how the consistency comes about in the two cases.

III. GENERAL CONSIDERATIONS FOR THE CONSISTENCY OF THE SELF-COUPLED FERMI THEORY

As discussed in Sec. I, the existence of a nonzero η^μ in a Lorentz-invariant theory places definite requirements on the structure of the function

$$C_\eta{}^{\mu\nu\lambda}(x)=i\langle0|[T^{\mu\nu}(x),j^\lambda(0)]|0\rangle.$$

In this section we study this requirement as it is related to the structures generated by the Lagrangian (2.1) in order to test the consistency of the solutions that we have found. The detailed calculation of $C_\eta^{\mu\nu\lambda}$ is fairly complex and is done in the next section, using the same techniques as in Sec. II.

This analysis is facilitated by noting, since the operators $T^{\mu\nu}(y)$ and $j^\lambda(x)$ are Hermitian, it follows that

$$2\epsilon(y^0-x^0)\ \mathrm{Re}\, i\langle 0|(T^{\mu\nu}(y)j^\lambda(x))_+|0\rangle = i\langle 0|[T^{\mu\nu}(y),j^\lambda(x)]|0\rangle = C_\eta^{\mu\nu\lambda}(y-x)\,. \quad (3.1)$$

We calculate the time-ordered product using the function

$$T_\eta^{\mu\nu\lambda}(y-x)=i\left[\frac{-i\delta}{\delta J^\lambda(x)}\frac{\langle 0\sigma_1|T^{\mu\nu}(y)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}+\frac{\langle 0\sigma_1|[i\delta/\delta J^\lambda(x)]T^{\mu\nu}(y)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}\right]_{J=0}$$
$$=i\left[\frac{\langle 0\sigma_1|(T^{\mu\nu}(x)j^\lambda(x))_+|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}-\frac{\langle 0\sigma_1|T^{\mu\nu}(y)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}\frac{\langle 0\sigma_1|j^\lambda(x)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}\right]_{J=0}. \quad (3.2)$$

Hence we have

$$C_\eta^{\mu\nu\lambda}(y-x)=2\epsilon(y^0-x^0)\ \mathrm{Re}\,T_\eta^{\mu\nu\lambda}(y-x) + \text{irrelevant constant terms.} \quad (3.3)$$

The extra terms in Eq. (3.3) are the terms involving the vacuum expectation values of $T^{\mu\nu}$ and j^λ separately in Eq. (3.2). These are position-independent constants when the external sources are turned off and do not contribute to the commutator $[J^{\mu\nu},j^\lambda]$. Through the use of the chain rule (2.10) we may rewrite (3.2) in the form

$$T_\eta^{\mu\nu\lambda}(y-x)\equiv \mathfrak{D}^{\lambda\beta}(x-z)T_{\dot\beta}^{\mu\nu}(z-y)/(2\pi)^4\,. \quad (3.4)$$

Retaining only the relevant terms we now find

$$C_\eta^{\mu\nu\lambda}(y-x)=2\epsilon(y^0-x^0)\ \mathrm{Re}\int d^4k \times e^{ik\cdot(y-x)}\mathfrak{D}^{\lambda\beta}(k)T_{\dot\beta}^{\mu\nu}(k)\,. \quad (3.5)$$

It is now possible to determine the exact tensor structure that $C_\eta^{\mu\nu\lambda}(k)$ will take, using either of the photon propagators of Sec. II. The considerations of this section depend only on the gauge of the photon propagator and are independent of perturbation theory. The all-important check of the coefficients of these structures within the framework of calculational procedures we have used will be performed in the next section.

We shall need the identities

$$\epsilon(x)\int d^4k\, e^{ik\cdot x}p\frac{1}{k^2+m^2}=\pi i\int d^4k\,\epsilon(k^0)\delta(k^2+m^2)e^{ik\cdot x}$$

and for η^μ light- or timelike

$$\epsilon(x)\int d^4k\, e^{ik\cdot x}p\frac{1}{\eta\cdot k}=\pi i\int d^4k\,\delta(\eta\cdot k)e^{ik\cdot x}.$$

The consistency of the theory must be determined by singularities of $\mathfrak{D}^{\lambda\beta}(k)$ so we exploit the facts that $T_{\dot\beta}^{\mu\nu}$ has the property $k^\beta[T_{\dot\beta}^{\mu\nu}(k)-T_{\dot\beta}^{\mu\nu}(0)]=0$ and is real for $-k^2<4m^2$. It then follows, using the Bjorken propagator with $\eta^2=0$ and only the parts of $T_\beta^{\mu\nu}(x)$ relevant to the consistency of the theory, that

$$C_\eta^{\mu\nu\lambda}(x)=\int d^4k\, e^{ik\cdot x}\left[-\frac{1}{(2\pi)^3Li}\bar g^{\lambda\beta}T_{\dot\beta}^{\mu\nu}(k)\epsilon(k^0)\delta(k^2)+\frac{\epsilon(x)}{(2\pi)^4C}p\left[\frac{k^\lambda k^\beta}{(\eta\cdot k)^2}\right]T_{\dot\beta}^{\mu\nu}(0)\right].$$

Exploiting the fact that integrals that enter into determining the consistency must respect conservation of energy momentum (or regularity requirements), it may be shown that $T_{\dot\beta}^{\mu\nu}(0)\propto\eta_\beta\eta^\mu\eta^\nu$. Finally, using the previously made observation that $\delta(k^2)\bar g^{\lambda\beta}\eta_\beta=\delta(k^2)\bar g^{\lambda\beta}k_\beta=0$, we may establish that $C_\eta^{\mu\nu\lambda}(x)$ has the form

$$C_\eta^{\mu\nu\lambda}(x)=\int d^4k\, e^{ik\cdot x}[(C_1/i)\epsilon(k^0)\delta(k^2)\eta\cdot k\bar g^{\lambda(\mu}k^{\nu)} + (C_2/i)\delta(\eta\cdot k)k^\lambda\eta^\mu\eta^\nu]\,. \quad (3.6)$$

From the arguments of Sec. I, it follows that $C_1-C_2=1/(2\pi)^3$. It is the burden of an explicit calculation to verify this relation. Note that the second term of (3.6) violates current conservation and is the explicit result of taking the cutoff extremely seriously. This term, as was pointed out in Sec. II, is directly connected with the zero-mass particle and nonvanishing η. However, it represents an anomalous transition between degenerate states and not a massless particle spectral weight.

We can now consider the case where the cutoff is not taken as seriously and the propagator is in the Lorentz gauge. It follows that

$$C_\eta^{\mu\nu\lambda}(x)=\int\frac{d^4k}{(2\pi)^3}e^{ik\cdot x}\left(\frac{6}{L}\right)i\left[g^{\lambda\beta}-\frac{k^\lambda k^\beta}{k^2}\right]\times\delta(k^2)\epsilon(k^0)T_{\dot\beta}^{\mu\nu}(k)\,.$$

From conservation of energy momentum we now must have $T_{\dot\beta}^{\mu\nu}(0)=0$. Energy-momentum conservation combined with regularity requirements show that the only form of $T_{\dot\beta}^{\mu\nu}(k)$ which can make a nonvanishing contribution to the consistency is $T_{\dot\beta}^{\mu\nu}(k)=C_3\eta^\beta\bar g^{\mu\nu}k^2$. Hence, the significant terms of $C_\eta^{\mu\nu\lambda}$ are

$$C_\eta^{\mu\nu\lambda}(x)=\int\frac{d^4k}{(2\pi)^3}e^{ik\cdot x}\frac{1}{i}\left(\frac{C_3 6}{L}\right)k^\lambda(\eta\cdot k)\bar g^{\mu\nu}\epsilon(k^0)\delta(k^2)\,.$$

Thus the gauge parts of $\mathfrak{D}^{\mu\nu}$ are completely responsible for consistency. This is one of the forms mentioned in Sec. I. Straightforward calculation shows that $C_3=\frac{1}{6}L$ and that there is no restriction on η^μ. Again we emphasize that the ultimate consistency of the theory hinges on the explicit calculation of C_3 within our perturbation scheme to verify that the above equation is satisfied identically.

IV. EXPLICIT CALCULATION OF $C_\eta^{\mu\nu\lambda}$

The symmetrical energy-momentum tensor for the Lagrangian (2.1) is

$$T^{\mu\nu}=-\tfrac{1}{4}g^{\mu\nu}g_0 j^\alpha j_\alpha-\tfrac{1}{4}i[\bar{\psi}\beta\gamma^{(\nu}\partial^{\mu)}\psi-\partial^{(\mu}\bar{\psi}\beta\gamma^{\nu)}\psi]. \quad (4.1)$$

It may be verified by direct calculation that $T^{\mu\nu}$ is transverse, i.e., $\partial_\mu T^{\mu\nu}=0$. By recalling that $\langle 0\sigma_1|j^\mu(y)|0\sigma_2\rangle/\langle 0\sigma_1|0\sigma_2\rangle=\eta^\mu(y)$, it is possible to evaluate the vacuum expectation values of $T^{\mu\nu}(y)$ in the presence of the source J. We find, neglecting terms of the form $-i(\delta\eta^\alpha/\delta J^\lambda)$ to be consistent with the approximations made in Sec. II for the function $G(x,y)$, that

$$\langle 0\sigma_1|T^{\mu\nu}(y)|0\sigma_2\rangle/\langle 0\sigma_1|0\sigma_2\rangle=-\tfrac{1}{4}g^{\mu\nu}g_0[\eta^\alpha(y)\eta_\alpha(y)]$$
$$-\tfrac{1}{4}[\partial_\xi{}^{(\mu}\operatorname{tr}[G(y,\xi)\gamma^{\nu)}]-\partial y^{(\mu}\operatorname{tr}[G(y\xi)\gamma^{\nu)}]_{\xi\to y}. \quad (4.2)$$

From the equations of Sec. II one may easily deduce that

$$(-i\delta/\delta J_\mu(z))\eta^\alpha(x)=-iG^{\mu\alpha}(z,x)$$
$$=\mathfrak{D}^{\mu\beta}(z,y)\operatorname{tr}[\gamma_\beta G(y,x)\gamma^\alpha G(x,y)], \quad (4.3a)$$

and that

$$(-i\delta/\delta J^\mu(z))G(x,y)$$
$$=-i\mathfrak{D}_\mu{}^\alpha(z,\xi)G(x,\xi)q\gamma_\alpha G(\xi,y). \quad (4.3b)$$

Inserting Eq. (4.2) into definition (3.2) and using Eqs. (4.3) we find, when $J=0$, that

$$T_\eta{}^{\mu\nu\lambda}(y-x)$$
$$=i\mathfrak{D}_\beta{}^\lambda(x-z)[-\tfrac{1}{2}g^{\mu\nu}g_0\eta^\alpha\operatorname{tr}\gamma^\beta G(z-y)\gamma_\alpha G(y-z)$$
$$-\tfrac{1}{4}\{\partial_\xi{}^{(\mu}\operatorname{tr}G(y-z)q\gamma^\beta G(z-\xi)\gamma^{\nu)}$$
$$-\partial_y{}^{(\mu}\operatorname{tr}G(y-z)q\gamma^\beta G(z-\xi)\gamma^{\nu)}\}_{\xi=y}]. \quad (4.4)$$

Taking the Fourier transform of this we find

$$T_\eta{}^{\mu\nu\lambda}(k)$$
$$=\mathfrak{D}_\beta{}^\lambda(k)\int d^4p[-\tfrac{1}{2}g^{\mu\nu}g_0 i\eta^\alpha(2\pi)^4\operatorname{tr}(\gamma^\beta G(p)\gamma_\alpha G(p+k))$$
$$-\tfrac{1}{4}i(2\pi)^4(\operatorname{tr}[\gamma^{(\nu}G(p)\gamma^\beta qG(p+k)])(2p^{\mu)}+k^{\mu)})]. \quad (4.5)$$

We have studied the structure of the first term of Eq. (4.5) in Sec. II. It is essentially the function $\pi^{\mu\nu}(k)$ which was given by Eq. (2.55). We have not encountered the third term before in the above form so it is convenient to make the definition

$$E^{\beta\mu\nu}(k)\equiv\tfrac{1}{4}(2\pi)^4 i\int d^4p$$
$$\times\operatorname{tr}[\gamma^{(\nu}G(p)\gamma^\beta qG(p+k)][2p^{\mu)}+k^{\nu)}]. \quad (4.6)$$

In terms of the functions $\pi^{\mu\nu}(k)$ and $E^{\beta\mu\nu}(k)$ we may write (4.5) in the compact form

$$T_\eta{}^{\mu\nu\lambda}(k)=\mathfrak{D}_\beta{}^\lambda(k)[\tfrac{1}{2}g^{\mu\nu}\eta^\alpha\pi_\alpha{}^\beta(k)-E^{\beta\mu\nu}(k)]. \quad (4.7)$$

We shall now analyze $E^{\beta\mu\nu}(k)$ in a manner modeled on the treatment given $\pi^{\mu\nu}(k)$ in Sec. II. The separation $G=(G_++G_-)/2+g(G_+-G_-)/2$ allows us to display $E^{\beta\mu\nu}(k)$ in the form

$$E^{\beta\mu\nu}(k)=\tfrac{1}{4}i(2\pi)^4\int d^4p[\operatorname{tr}\gamma^{(\nu}G_+(p)\gamma^\beta G_+(p+k)$$
$$-\operatorname{tr}\gamma^{(\nu}G_-(p)\gamma^\beta G_-(p+k)][2p^{\mu)}+k^{\mu)}]. \quad (4.8)$$

From this form it is easily seen that $E^{\beta\mu\nu}(k)$ is odd in η and even in k. It is further seen that $E^{\beta\mu\nu}(k)$ diverges quadratically as the momentum cutoff Λ increases. We must exercise the usual caution in translating integrals if we wish to use the cutoff in the manner used to determine the Bjorken form of the propagator. If we do not exercise this caution we drop the strange extraneous finite terms which are important in this form of the theory. It can be seen at once that we shall encounter difficulties in this formulation. From the analysis of Sec. III it follows that the exact determination of

$$E^{\beta\mu\nu}(0)=\tfrac{1}{4}(2\pi)^4 i\int d^4p\operatorname{tr}[q\gamma^{(\nu}G(p)\gamma^\beta G(p)]p^{\mu)}$$

is essential. This quantity is, however, not related to $\operatorname{tr}qG(x,x)$, the standard divergent quantity of the theory by a Ward's identity. A direct calculation of this quantity cannot really be regarded too seriously for this reason. Thus the only reasonable procedure for handling $E^{\beta\mu\nu}(0)$ is to adjust it so as to guarantee the consistency of the theory. Thus the Bjorken formulation of the theory is consistent by a bit of trickery which destroys the beauty of regarding it as a theory with a cutoff which can be taken literally. If $\eta^2\neq 0$, we have finished the calculation and must conclude that all consistency proceeds through the anomalous $\delta(\eta\cdot k)$ term. This is essentially the case for the more reasonable restriction $\eta^2=0$. To see this we determine

$$C_\eta{}'^{\mu\nu\lambda}(k)\equiv i(6/L)(\epsilon(k^0)\delta(k^2)/(2\pi)^4)\bar{g}^{\lambda\beta}$$
$$\times[T_{\dot\beta}{}^{\mu\nu}(k)-T_{\dot\beta}{}^{\mu\nu}(0)]$$
$$=(6/L)(\delta(k^2)/(2\pi)^4 i)\bar{g}^{\lambda\beta}\epsilon(k^0)$$
$$\times[-E_{\dot\beta}{}^{\mu\nu}(0)+E_{\dot\beta}{}^{\mu\nu}(k)].$$

A tedious and boring calculation shows that

$$C_\eta{}'^{\mu\nu\lambda}(k)=(F/L)(\pi i/(2\pi)^4)(\eta\cdot k)\bar{g}^{\lambda(\mu}k^{\nu)}\delta(k^2)\epsilon(k^0).$$

In the limit that $\Lambda \gg m$, $C_\eta'^{\mu\nu\lambda}(k)=0$, and hence we finally find that in the Bjorken formulation for large Λ that

$$C_\eta{}^{\mu\nu\lambda}(k)=(\delta(\eta\cdot k)/(2\pi)^3 i)k^\lambda\eta^\mu\eta^\nu.$$

Thus, the consistency hinges on the current conservation-violating aspect of the theory. Spurious states guarantee this consistency. Note that they are very complex in origin and related not only to the broken Lorentz symmetry and the zero-mass particle but also to the nature of the approximation scheme.

The consistency with the Lorentz-gauge photon propagator proceeds with ease. The most divergent terms which come from $T_\beta{}^{\mu\nu}(0)$ and hence $E^{\beta\mu\nu}(0)$ must not contribute in accord with the discussion in Sec. II. In addition, we are not interested in any of the anomalous terms which come from taking the cutoff literally. Thus we may let Λ/m become large relative to $g_0\eta^\alpha$ and translate in η^α the integrals which appear in $E^{\beta\mu\nu}(k)$. Performing the translation we find from (4.8) that

$$\begin{aligned}E^{\beta\mu\nu}(k)=&\tfrac{1}{4}(2\pi)^4 i\int d^4p\ \mathrm{tr}[\gamma^{(\nu}G(p)\gamma^\beta G(p+k)]\\ &\times[2(p^{\mu)}-g_0\eta^{\mu)})+k^{\mu)}-2(p^{\mu)}+g_0\eta^{\mu)})-k^{\mu)}]\\ &=\eta^{(\mu}\pi^{\nu)\beta}(k).\end{aligned}$$

With this, Eq. (4.7) becomes

$$T_\eta{}^{\mu\nu\lambda}(k)=\mathfrak{D}_\beta{}^\lambda(k)[\tfrac{1}{2}g^{\mu\nu}\eta^\alpha\pi_\alpha{}^\beta(k)-\eta^{(\mu}\pi^{\nu)\beta}(k)].$$

Inserting $\pi^{\alpha\beta}(k)$ as given by Eq. (2.55) into the above expression we find, after application of (3.3), that

$$C_\eta{}^{\mu\nu\lambda}(k)=\frac{k^\lambda(\eta\cdot k)}{(2\pi)^3 i}\bar{g}^{\mu\nu}\epsilon(k^0)\delta(k^2).$$

This is exactly the correct form to guarantee consistency.

V. CONCLUSIONS

We conclude that the Bjorken theory can be formulated in a consistent manner. The original form of this theory is somewhat objectionable because of its violation of charge conservation. This violation is, however, not serious since it is manifested entirely in the gauge structure of the resulting electrodynamics. Because of it the theory is consistent in a spurious manner. It is primarily because of this property that this formulation is of interest. In a more realistic calculation in which the cutoff of the theory is viewed in a normal perspective, all calculation proceeds in a natural and unstrained manner.

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